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Solving Delay Differential Equations Using the Method of Steps

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Abstract: **Delay differential equations (DDEs) are a class of differential equations where the derivative of the unknown function at a given time depends not only on the current state but also on its values at previous times. These equations arise in numerous fields, including biology, engineering, and economics. Unlike ordinary differential equations (ODEs), DDEs incorporate time delays, making their solution more complex due to the need for a history function. A common form of a first-order linear DDE with a constant delay is:**

$$\frac{dy(t)}{dt} = -ay(t) + by(t-\tau),$$

where $y(t)$ is the unknown function, a and b are constants, and $\tau > 0$ is the time delay. Solving DDEs analytically or numerically requires specialized techniques, and one of the most straightforward and intuitive methods for constant-delay DDEs is the method of steps. This approach breaks the problem into sequential intervals, solving an ODE in each interval using the solution from the previous interval as a history function. In this article, we explore the method of steps in detail, including its mathematical derivation, assumptions, practical implementation, and an illustrative example.

Keywords: **Delay differential equations (DDEs), numerous fields, including biology, engineering, economics.**

1. UNDERSTANDING DELAY DIFFERENTIAL EQUATIONS

DDEs generalize ODEs by introducing time delays, which reflect real-world phenomena where effects are not instantaneous. For example:

- In **biology**, the growth rate of a population may depend on its size at a previous time due to gestation or maturation periods.
- In **control systems**, feedback loops often involve delays due to signal processing or actuator response times.
- In **epidemiology**, disease spread models may account for incubation periods.

To solve a DDE like:

$$\frac{dy(t)}{dt} = -ay(t) + by(t-\tau), t \geq 0,$$

we need:

- An **initial condition** over a time interval, typically a history function $y(t) = \phi(t)$ for $t \in [-\tau, 0]$, since the solution at $t > 0$ depends on past values.
- The delay τ, which is constant in this case (variable delays require more advanced methods).
- Parameters a and b, which determine the system's dynamics.

The method of steps is particularly suited for DDEs with constant delays and continuous history functions, as it leverages the delay to solve the equation iteratively over successive time intervals.

2. THE METHOD OF STEPS: CONCEPTUAL OVERVIEW

The method of steps exploits the constant delay τ to transform the DDE into a series of ODEs. The key idea is to solve the DDE over intervals of length τ, using the known history function for $t \in [-\tau, 0]$ to compute the solution for $t \in [0, \tau]$, then using that solution as the history for the next interval $t \in [\tau, 2\tau]$, and so on. In each interval, the term $y(t - \tau)$ becomes a known function, reducing the DDE to an ODE that can be solved using standard techniques. This iterative process continues until the desired time range is covered.

The method assumes:

- The delay τ is constant.
- The history function $\phi(t)$is continuous on $[-\tau, 0]$ to ensure a smooth solution.
- The DDE is linear or can be approximated linearly in each interval.

While simple in concept, the method of steps can become computationally intensive for large time ranges, as the solution's complexity grows with each interval.

Step-by-Step Derivation

Consider the linear DDE:

$$\frac{dy(t)}{dt} = -ay(t) + by(t-\tau), t \geq 0,$$

with history function $y(t) = \phi(t)$ for $t \in [-\tau, 0]$. Let's apply the method of steps to solve this equation.

- **First Interval:** $t \in [0, \tau]$

For $t \in [0, \tau]$, the delayed term $y(t - \tau)$ refers to times $t - \tau \in [-\tau, 0]$, where the solution is given by the history function $\phi(t - \tau)$. The DDE becomes:

$$\frac{dy(t)}{dt} = -ay(t) + b\phi(t-\tau).$$

This is a first-order linear ODE of the form:

$$\frac{dy(t)}{dt} + ay = b\phi(t-\tau).$$

The integrating factor is$e^{\int a dt} = e^{at}$. Multiply through by e^{at}:

$$e^{at}\frac{dy}{dt} + ae^{at}y = be^{at}\phi(t-\tau).$$

The left-hand side is the derivative of a product:

$$\frac{d}{dt}(ye^{at}) = be^{at}\phi(t-\tau).$$

Integrate from 0 to t:

$$y(t)e^{at} - y(0)e^{a.0} = \int_0^t be^{as}\phi(s-\tau)ds.$$

Since $y(0) = \phi(0)$, we have:

$$y(t) = e^{-at}\phi(0) + be^{-at}\int_0^t be^{as}\phi(s-\tau)ds.$$

This gives $y(t)$ for $t \in [0, \tau]$, which we denote as $y_1(t)$.

- **Second Interval:** t∈[τ,2τ]

For $t \in [\tau, 2\tau]$, the delayed term $y(t-\tau)$ refers to $t-\tau \in [0,\tau]$, where the solution is $y_1(t-\tau)$. The DDE becomes:

$$\frac{dy(t)}{dt} = -ay(t) + by_1(t-\tau).$$

This is another ODE:

$$\frac{dy}{dt} + ay = by_1(t-\tau).$$

Using the same integrating factor e^{at}:

$$\frac{d}{dt}(ye^{at}) = be^{at}y_1(t-\tau).$$

Integrate from τ to t:

$$y(t)e^{at} - y(\tau)e^{a\tau} = \int_\tau^t be^{as}y_1(s-\tau)ds.$$

Here, $y(\tau) = y_1(\tau)$, and $y_1(t-\tau)$ is known from the first interval. Solving gives:

$$y(t) = e^{-a(t-\tau)}y_1(\tau) + be^{-at}\int_\tau^t be^{as}y_1(s-\tau)ds.$$

This yields $y_2(t)$ for $t \in [\tau, 2\tau]$.

- **General Interval:** $t \in [n\tau, (n+1)\tau]$

For the $n-th$ interval, $y(t-\tau)$ is given by the solution $y_{n-1}(t-\tau)$ from the previous interval. The ODE is:

$$\frac{dy(t)}{dt} = -ay(t) + by_{n-1}(t-\tau).$$

The solution is:

$$y_n(t) = e^{-a(t-n\tau)}y_{n-1}(n\tau) + be^{-at}\int_{n\tau}^t e^{as}y_{n-1}(s-\tau)ds.$$

Continuity at $t = n\tau$ ensures $y_n(n\tau) = y_{n-1}(n\tau)$.

Example Application: Population Model with Delay

Consider a DDE modeling population growth with a maturation delay:

$$\frac{dy(t)}{dt} = -0.5y(t) + 0.3y(t-1), t \geq 0,$$

with history function $y(t) = 1$for $t \in [-1,0]$. We solve using the method of steps.

- **Interval** $t \in [0,1]$:

Here, $y(t-1) = 1$, so:

$$\frac{dy}{dt} = -0.5y + 0.3 \cdot 1 = -0.5y + 0.3.$$

Solve the ODE:

$$\frac{dy}{dt} + 0.5y = 0.3.$$

Integrating factor: $e^{\int 0.5dt} = e^{0.5t}$. Multiply through:

$$\frac{d}{dt}(ye^{0.5t}) = 0.3e^{0.5t}.$$

Integrate from 0 to t:

$$ye^{0.5t} - y(0) = \int_0^t 0.3e^{0.3s} ds = 0.3 \cdot \frac{e^{0.5t} - 1}{0.5} = 0.6(e^{0.5t} - 1).$$

Since $y(0) = 1$:

$$y(t) = e^{0.5t} + 0.6(1 - e^{0.5t}) = 0.6 + 0.4e^{0.5t}.$$

This is $y_1(t)$for $t \in [0,1]$.

- **Interval** $t \in [1,2]$:

Now, $y(t-1) = y_1(t-1) = 0.6 + 0.4e^{0.5(t-1)}$. The ODE is:

$$\frac{dy}{dt} = -0.5y + 0.3(0.6 + 0.4e^{0.5(t-1)})$$

This is more complex, but solvable similarly. The process continues for further intervals, with each solution feeding into the next.

3. ADVANTAGES AND LIMITATIONS

The method of steps is intuitive and exact for linear DDEs with constant delays. It is particularly effective for short time ranges or when an analytical solution is desired. However, it has drawbacks:

- **Complexity Growth**: Solutions become increasingly complicated with each interval, as integrals involve previous solutions.
- **Numerical Preference**: For long time spans, numerical methods (e.g., Runge-Kutta adapted for DDEs) are often more practical.
- **Constant Delays**: The method is less suited for variable delays or distributed delays.

Alternative Methods

Other approaches for solving DDEs include:

- **Numerical Methods**: Specialized solvers like dde23 in MATLAB or implicit Runge-Kutta methods.
- **Laplace Transforms**: Useful for linear DDEs, transforming the equation into an algebraic form.
- **Asymptotic Analysis**: For small or large delays, perturbation methods can approximate solutions.
- **Characteristic Equation**: Analyzes stability and long-term behavior by solving the characteristic equation.

4. CONCLUSION

The method of steps provides a systematic approach to solving delay differential equations with constant delays, converting a complex DDE into a sequence of ODEs. Its iterative nature makes it accessible for analytical solutions over short intervals, with applications in modeling delayed phenomena across disciplines. While limited by growing complexity, it remains a valuable tool for understanding DDE dynamics, complementing numerical and transform-based methods.

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